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# The Effect of Spatial Discretization on the Steady-State and Transient Solutions of a Dispersive Wave Equation

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The effect of replacing the spatial derivatives in a dispersive wave equation with second-order centered finite differences is examined with the use of Fourier transform techniques. The discretization is shown to both decrease the rate of spatial decay of the steady-state solution, and to introduce additional new transients at least as persistent as those in the differential case.

## 1. INTRODUCTION

The central issue in approximating a partial differential equation by a finite difference equation is the degree to which the difference equation solution agrees with the solution to the differential equation. This agreement can be considered in both its quantitative (e.g., relative error), or qualitative aspects, e.g., behavior of transients, propagation of fronts, etc.

In this paper we investigate a prototype dispersive wave (hyperbolic) equation. Our particular example arose in numerical weather prediction. Meteorologists speak of the "geostrophic balance" that exists in the atmosphere between Coriolis force and pressure gradient force. When the atmosphere is modeled by the so-called meteorological primitive equations, this balance condition is contained in a dispersive wave equation. This dispersive wave mechanism is the primary one by which the model atmosphere reacts to imbalances (either numerical or due to errors in observational data) in initial conditions, and moves toward the geostrophic balance in the process referred to as geostrophic adjustment. Forecasts at times before this balance is reached are generally inaccurate.

Our model arose in the simplest special case of this geostrophic adjustment process. Since all practical solutions of the primitive equations are numerical, our purpose is to consider the effect of quasi-discretization on the solution of the dispersive wave equation. The quasi-discretization chosen is the most common method used in practice—centered, second-order spatial differences. The basic model is linearized, since we wish to have a solution to the differential case to compare the quasi-discrete solution to. The comparison of solutions comes in two important areas: the dependence of the final state on the initial data, and the order of magnitude of the transients involved.





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In both cases solutions are obtained by Fourier transform methods, with steady-state solutions extracted directly from the transforms and inverted in closed form, and the asymptotic behavior of the transients determined by the method of stationary phase.

Our analysis shows that discretization of spatial derivatives has two major effects:

(1) The initial conditions of the differential problem contribute to the steady-state solution in a manner that decays exponentially with distance. This qualitative effect is retained in the quasi-discrete formulation, but the rate of decay is decreased.

(2) The discretization introduces additional transients beyond those encountered in the differential case. These new transients are at least of the same magnitude of decay as the differential transients, and, in some instances, they are more persistent.

## 2. The Differential One-Dimensional Adjustment Process

One of the simplest models of dispersive waves is the linearized one-dimensional shallow water equations with no mean flow, in an infinite region:

$$(\partial u/\partial t) - fv + g(\partial h/\partial x) = 0; \tag{1}$$

$$(\partial v/\partial t) + fu = 0; \tag{2}$$

$$(\partial h/\partial t) + H(\partial u/\partial x) = 0; \tag{3}$$

where $u$ is the perturbation velocity in the $x$ direction, $v$ is the perturbation velocity normal to the $x$ direction, $H$ and $h$ are the mean and perturbed heights of the free surface, respectively, and $g > 0$ and $f > 0$ are gravitational and Coriolis parameters, respectively. This model is especially important in the study of the meteorological problem of geostrophic adjustment, and has been studied in some detail by Rossby [1], Cahn [2], Blumen [3], and Winninghoff [4]. In their papers, the model has been studied by eliminating between the equations to arrive at

$$(\partial^2 u/\partial t^2) + f^2 u - gH(\partial^2 u/\partial x^2) = 0, \tag{4}$$

then solving (4) by a Fourier transform approach. After solving (4), solutions for $h$ and $v$ are obtained by substitution into (2) and (3), although closed-form solutions are not produced in some of the papers. Note that the dispersive character of (4) is clearly seen by assuming a wave solution

$$u(x, t) = Ae^{i(kx - \nu t)},$$

which leads immediately to

$$\nu^2 = f^2 + k^2 gH = f^2(1 + \lambda^2 k^2), \qquad \lambda = (gH)^{1/2}/f. \tag{5}$$

An alternative means of solving (1)–(3) which is superior in that it does not require elimination, it produces $u$, $v$, and $h$ without back substitution, it yields interesting insights into the transient and steady-state behavior of the solutions in the differential case, and it has an extension in the quasi-discrete case that is quite illuminating, is to directly transform the system (1)–(3). Thus, if we denote Fourier transforms by an overlying tilde, e.g.,

$$\mathscr{F}\{u\} = \tilde{u}(k, t) = \int_{-\infty}^{\infty} u(x, t)\, e^{-ikx}\, dx,$$

etc., then (1)–(3) reduce to

$$d\tilde{u}/dt = f\tilde{v} - ikg\tilde{h}, \tag{6}$$

$$d\tilde{v}/dt = -f\tilde{u}, \tag{7}$$

$$d\tilde{h}/dt = -ikH\tilde{u}, \tag{8}$$

together with initial conditions

$$\tilde{u}_0 = \tilde{u}(k, 0) = \int_{-\infty}^{\infty} u(x, 0)\, e^{-ikx}\, dx, \tag{9}$$

etc. Since (6)–(8) are a coupled set of constant coefficient ordinary differential equations, they can be solved by the usual process of finding the eigenvalues and eigenvectors of the coefficient matrix. This leads straightforwardly to

$$\begin{aligned} \tilde{u} &= (\alpha_2/\nu)\, e^{i\nu t} - (\alpha_3/\nu)\, e^{-i\nu t}, \\ \tilde{v} &= ikg\alpha_1 + (if/\nu^2)\, \alpha_2 e^{i\nu t} - (if/\nu^2)\, \alpha_3 e^{-i\nu t}, \\ \tilde{h} &= f\alpha_1 - (kH/\nu^2)\, \alpha_2 e^{i\nu t} - (kH/\nu^2)\, \alpha_3 e^{-i\nu t}, \end{aligned} \tag{10}$$

where $\nu$ is given by (5), and the $\alpha_i$ are picked to satisfy the initial conditions. Observe that the $e^{i\nu t}$ and $e^{-i\nu t}$ terms both represent the transforms of transients in the time domain. Solving the initial conditions for the $\alpha_i$, collecting like terms in (10), and simplifying yields

$$\begin{aligned} \tilde{u}(k, t) &= \tilde{u}_0 \cos \nu t + \frac{f\tilde{v}_0}{\nu} \sin \nu t - \frac{ikg\tilde{h}_0}{\nu} \sin \nu t, \\ \tilde{v}(k, t) &= -\frac{f}{\nu}\, \tilde{u}_0 \sin \nu t + \left\{\frac{k^2gH}{\nu^2} + \frac{f^2}{\nu^2} \cos \nu t\right\} \tilde{v}_0 + \frac{ikgf}{\nu^2}\{1 - \cos \nu t\}\, \tilde{h}_0, \\ \tilde{h}(k, t) &= -\frac{ikH}{\nu}\, \tilde{u}_0 \sin \nu t - \frac{ikHf}{\nu^2}\{1 - \cos \nu t\}\, \tilde{v}_0 + \left\{\frac{f^2}{\nu^2} + \frac{k^2gH}{\nu^2} \cos \nu t\right\} \tilde{h}_0 . \end{aligned} \tag{11}$$

We observe that (11) immediately yields by inspection the transform of the steady-state (often referred to by meteorologists as the "balanced" state) solution:

$$\begin{aligned} \tilde{u}_S(k) &= 0, \\ \tilde{v}_S(k) &= \frac{k^2gH}{\nu^2}\, \tilde{v}_0 + \frac{ikgf}{\nu^2}\, \tilde{h}_0 = \tilde{v}_0 + \frac{f^2}{\nu^2} \left\{\frac{ikg}{f}\, \tilde{h}_0 - \tilde{v}_0\right\}, \\ \tilde{h}_S(k) &= -\frac{ikHf}{\nu^2}\, \tilde{v}_0 + \frac{f^2}{\nu^2}\, \tilde{h}_0 = \tilde{h}_0 + \frac{H}{f} \cdot \frac{f^2}{\nu^2}\, ik \left\{\frac{ikg}{f}\, \tilde{h}_0 - \tilde{v}_0\right\}. \end{aligned} \tag{12}$$

These can be immediately inverted by the convolution theorem, and the observation that

$$\int_{-\infty}^{\infty} e^{-|x|/\lambda} e^{-ikx}\, dx = 2\lambda/(1 + k^2\lambda^2) = (2\lambda/\nu^2) f^2$$

to yield

$$\begin{aligned} u_S(x) &= 0, \\ v_S(x) &= v(x, 0) + \frac{1}{2\lambda} \int_{-\infty}^{\infty} e^{-|x-s|/\lambda} \left\{ \frac{g}{f} \frac{\partial h}{\partial x} (s, 0) - v(s, 0) \right\} ds, \\ h_S(x) &= h(x, 0) - \frac{H}{2\lambda^2 f} \int_{-\infty}^{\infty} e^{-|x-s|/\lambda} \left\{ \frac{g}{f} \frac{\partial h}{\partial x} (s, 0) - v(s, 0) \right\} ds. \end{aligned} \tag{13}$$

We observe, in passing: (a) $\tilde{u}(k, 0)$ does not contribute to any of the steady-state solutions. (b) The term in (13),

$$\{-(g/f) \frac{\partial h}{\partial x} (s, 0) + v(s, 0)\}, \tag{14}$$

is the initial value of the quantity usually referred to in meteorology as the ageostrophic wind. It can be considered as a measure of "imbalance" in the initial state that contributes to the final state, and at steady state its effects are strongest in the immediate neighborhood of the initial imbalance and die off exponentially in space away from it.

Observe also that the transient part of (11) can be written

$$\begin{aligned} \tilde{u}_T(k, t) &= \tilde{u}_0 \cos \nu t + \frac{f}{\nu} \left\{ \tilde{v}_0 - \frac{ikg}{f} \tilde{h}_0 \right\} \sin \nu t, \\ \tilde{v}_T(k, t) &= - \frac{f}{\nu} \tilde{u}_0 \sin \nu t + \frac{f^2}{\nu^2} \left\{ \tilde{v}_0 - \frac{ikg}{f} \tilde{h}_0 \right\} \cos \nu t, \\ \tilde{h}_T(k, t) &= - \frac{ikH}{\nu} \tilde{u}_0 \sin \nu t + \frac{ikHf}{\nu^2} \left\{ \tilde{v}_0 - \frac{ikg}{f} \tilde{h}_0 \right\} \cos \nu t. \end{aligned} \tag{15}$$

Explicit inverses to these transforms are expressible in terms of convolutions involving $J_0[(1/\lambda)(\lambda^2 f^2 t^2 - x^2)^{1/2}]$. However, a simplified view of the asymptotic behavior of the transients is possibly by using the method of stationary phase. Let

$$d(x, t) = v(x, t) - (g/f) \frac{\partial h}{\partial x} (x, t).$$

Then

$$u_T(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(k, 0) \cos \nu t\, e^{ikx}\, dk + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f}{\nu} \tilde{d}(k, 0) \sin \nu t\, e^{ikx}\, dk. \tag{16}$$

Thus, using the method of stationary phase (Appendix 1), we can show, for fixed $x$ as $t \to \infty$:

$$u_T(x, t) \sim \left[\frac{\lambda f^2 t^2}{2\pi(\lambda^2 f^2 t^2 - x^2)^{3/2}}\right]^{1/2} \left\{ \left| \tilde{u}\left(\frac{\lambda^{-1}x}{(\lambda^2 f^2 t^2 - x^2)^{1/2}}, 0\right) \right| \right.$$
$$\times \cos\left(\frac{1}{\lambda}(\lambda^2 f^2 t^2 - x^2)^{1/2} + \phi_k\right) + \frac{(\lambda^2 f^2 t^2 - x^2)^{1/2}}{\lambda f t}$$
$$\left. \times \left| \tilde{d}\left(\frac{\lambda^{-1}x}{(\lambda^2 f^2 t^2 - x^2)^{1/2}}, 0\right) \right| \sin\left(\frac{1}{\lambda}(\lambda^2 f^2 t^2 - x^2)^{1/2} + \psi_k\right) \right\}, \quad (17)$$

where $\phi_k$ and $\psi_k$ are slowly varying phases. When $|x| \ll \lambda f t$, this is more conveniently approximated as

$$u_T(x, t) \sim \left[\frac{1}{2\pi\lambda^2 f t}\right]^{1/2} \left\{ \left| \tilde{u}\left(\frac{x}{\lambda^2 f t}, 0\right) \right| \cos(ft + \phi_k) + \left| \tilde{d}\left(\frac{x}{\lambda^2 f t}, 0\right) \right| \sin(ft + \psi_k) \right\}. \quad (18)$$

Thus the decay to steady state of $u$ at a fixed point $x$ is governed by two factors: (a) a $t^{-1/2}$ decay, which can be interpreted as the effect due to the dispersive nature of the process; and (b) an additional possible decay, depending on whether

$$\lim_{t\to\infty} |\tilde{u}(x/(\lambda^2 f t), 0)| = 0 \qquad \text{and} \qquad \lim_{t\to\infty} |\tilde{d}(x/(\lambda^2 f t), 0)| = 0.$$

Since

$$\lim_{t\to\infty} |\tilde{u}(x/(\lambda^2 f t), 0)| = \lim_{k\to 0} |\tilde{u}(k, 0)|,$$

this factor depends on the distribution into the longer waves of the initial values. Similar analyses of $v_T(x, t)$ and $h_T(x, t)$ yield

$$v_T(x, t) \sim \left[\frac{1}{2\pi\, \lambda^2 f t}\right]^{1/2} \left\{ - \left| \tilde{u}\left(\frac{x}{\lambda^2 f t}, 0\right) \right| \sin(ft + \phi_k) \right.$$
$$\left. + \left| \tilde{d}\left(\frac{x}{\lambda^2 f t}, 0\right) \right| \cos(ft + \psi_k) \right\}, \quad (19)$$

and

$$h_T(x, t) \sim \left[\frac{1}{2\pi\, \lambda^2 f t}\right]^{1/2} \left\{ - \frac{Hx}{\lambda^2 f^2 t} \left| \tilde{u}\left(\frac{x}{\lambda^2 f t}, 0\right) \right| \sin(ft + \phi_k) \right.$$
$$\left. + \frac{Hx}{\lambda^2 f^2 t} \left| \tilde{d}\left(\frac{x}{\lambda^2 f t}, 0\right) \right| \cos(ft + \psi_k) \right\}. \quad (20)$$

Note the one difference in $h_T(x, t)$. Due to the presence of the $(ik)$ term in $h_T(k, t)$, the decay due to dispersion of $h_T(x, t)$ proceeds as $t^{-3/2}$, rather than $t^{-1/2}$.

In summary then, the differential formulation of the dispersive wave model (1)–(3), is solved by transforming to a system of ordinary differential equations. The differential model always tends to a steady state, whose difference from the initial state is determined solely by what we have called the initial ageostrophic wind field, (14). The contributions from regions where this ageostrophic field is initially nonzero die out in the steady state exponentially with distance. Lastly, both the initial $u$ and ageostrophic wind fields contribute to transients that die out in time as $t^{-1/2}$ for the velocities, and $t^{-3/2}$ for the free surface height.

## 3. Second-Order Centered Finite Differences: The Quasi-Discrete Adjustment

In this section we present an analytic treatment of the most common difference scheme used for (1)–(3), and investigate the resulting effect on the steady-state and transient behavior discussed above.

Consider the continuous, quasi-discrete, second-order centered-leapfrog formulation of (1)–(3):

$$\begin{aligned} \partial u/\partial t &= fv(x,t) - (g/2\Delta x)[h(x+\Delta x,t) - h(x-\Delta x,t)], \\ \partial v/\partial t &= -fu(x,t), \\ \partial h/\partial t &= -(H/2\Delta x)[u(x+\Delta x,t) - u(x-\Delta x,t)]. \end{aligned} \tag{21}$$

If we Fourier transform this system, noting

$$\int_{-\infty}^{\infty} u(x+\Delta x,t)\, e^{-ikx}\, dx = e^{ik\Delta x}\tilde{u}(k,t),$$

we have

$$\begin{aligned} d\tilde{u}/dt &= f\tilde{v} - (ig/\Delta x)(\sin k\,\Delta x)\tilde{h}, \\ d\tilde{v}/dt &= -f\tilde{u}, \\ d\tilde{h}/dt &= -(iH/\Delta x)(\sin k\,\Delta x)\tilde{u}. \end{aligned} \tag{22}$$

Observe that this system is identical to (6)–(8) with $k$ replaced by

$$\sigma = (\sin k\,\Delta x)/\Delta x. \tag{23}$$

Thus the solution to (22) becomes identical to (11), except $k$ is replaced by $\sigma$, and $\nu$ by

$$\hat{\nu} = [1 + (\lambda/\Delta x)^2 \sin^2 k\,\Delta x]^{1/2}. \tag{24}$$

Observe that the sinusoidal terms in (11) continue to represent transients when $\nu$ is replaced by $\hat{\nu}$ and $k$ by $\sigma$. Thus the quasi-discrete case will tend toward a steady state whose transform is

$$\begin{aligned} \tilde{u}_S(k) &= 0, \\ \tilde{v}_S(k) &= \tilde{v}_0 + [1/(1+\lambda^2\sigma^2)]\{(i\sigma g/f)\,\tilde{h}_0 - \tilde{v}_0\}, \\ \tilde{h}_S(k) &= \tilde{h}_0 + (H/f)[1/(1+\lambda^2\sigma^2)]\, i\sigma\{(i\sigma g/f)\,\tilde{h}_0 - \tilde{v}_0\}. \end{aligned} \tag{25}$$

It is easily shown that

$$\frac{i\sigma g}{f}\tilde{h}_0 - \tilde{v}_0 = \mathscr{F}\left\{\frac{g}{f}\,\frac{h(x+\Delta x, 0) - h(x-\Delta x, 0)}{2\Delta x} - v(x, 0)\right\}, \tag{26}$$

where $\mathscr{F}\{\ \}$ denotes the Fourier transform. We shall let

$$\hat{d}(x, t) = v(x, t) - \frac{g}{f}\,\frac{h(x+\Delta x, t) - h(x-\Delta x, t)}{2\Delta x}. \tag{27}$$

Thus if we can invert $i\sigma/(1+\lambda^2\sigma^2)$ and $1/(1+\lambda^2\sigma^2)$, the steady-state solution would be available by convolution. These are inverted in Appendix 2, where we show

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+\lambda^2\sigma^2}\,e^{ikx}\,dk = \frac{\Delta x\, e^{-\beta|x|/\lambda}}{(\lambda^2+(\Delta x)^2)^{1/2}}\sum_{n=-\infty}^{\infty}\delta(x - 2n\,\Delta x),$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i\sigma}{1+\lambda^2\sigma^2}\,e^{ikx}\,dk = \frac{-\Delta x\, e^{-\beta|x|/\lambda}}{\lambda[\lambda^2+(\Delta x)^2]^{1/2}}\sum_{n=-\infty}^{\infty}\delta(x - (2n-1)\,\Delta x),$$

where

$$\beta = (\lambda/\Delta x)\sinh^{-1}(\Delta x/\lambda), \tag{28}$$

and $0 < \beta \leqslant 1$ with equality only for $\Delta x = 0$.

Thus we can arrive at

$$\begin{aligned} v_S(x) &= v(x, 0) - \Delta x\int_{-\infty}^{\infty}\left\{\frac{e^{-\beta|s|/\lambda}}{[\lambda^2+(\Delta x)^2]^{1/2}}\sum_{-\infty}^{\infty}\delta(s - 2n\,\Delta x)\right\}\hat{d}(x-s, 0)\,ds \\ &= v(x, 0) - \frac{1}{2[\lambda^2+(\Delta x)^2]^{1/2}}\sum_{n=-\infty}^{\infty}\{e^{-\beta|2n\Delta x|/\lambda}\,\hat{d}(x - 2n\,\Delta x, 0)(2\Delta x)\}, \end{aligned} \tag{29}$$

and similarly

$$\begin{aligned} h_S(x) &= h(x, 0) + \frac{H}{2\lambda[\lambda^2+(\Delta x)^2 f]^{1/2}} \\ &\quad\times\sum_{n=-\infty}^{\infty}\{e^{-\beta|(2n-1)\Delta x|/\lambda}\,\hat{d}(x - (2n-1)\,\Delta x, 0)(2\Delta x)\}. \end{aligned} \tag{30}$$

Clearly (29)–(30) tend toward the corresponding integral forms (13) as $\Delta x \to 0$.

These expressions can now be examined versus (13). The conclusions that we can draw are that conversion of (1)–(3) to centered, second-order, quasi-discrete form results in:

(1) The measure of imbalance, the initial ageostrophic wind, is converted to a finite differenced measure.

(2) Although at steady state the effects of an initial disturbance die off exponentially away from its original neighborhood, the rate of decay is less than in the differential case. Furthermore, the rate of decay decreases as $\Delta x/\lambda$ increases.

(3) For the steady states, only the values of the ageostrophic wind at alternating points are considered.

Our interpretation of why the sums in (29)–(30) involve only alternating points is that, when elimination is tried on (21), one ends with the equation for $u$ involving only alternating points:

$$\frac{\partial^2}{\partial t^2} u(x, t) = -f^2 u(x, t) + \frac{gH}{4(\Delta x)^2}\{u(x + 2\Delta x, t) - 2u(x, t) + u(x - 2\Delta x, t)\}. \quad (31)$$

A complete stationary phase analysis of the transient solution in the quasi-discrete case,

$$\begin{aligned}
\tilde{u}_T(k, t) &= \tilde{u}_0 \cos \hat{\nu} t + \frac{f}{\hat{\nu}}\left\{\tilde{v}_0 - \frac{i\sigma g}{f}\tilde{h}_0\right\} \sin \hat{\nu} t, \\
\tilde{v}_T(k, t) &= \frac{f}{\hat{\nu}}\tilde{u}_0 \sin \hat{\nu} t + \frac{f^2}{\hat{\nu}^2}\left\{\tilde{v}_0 - \frac{i\sigma g}{f}\tilde{h}_0\right\} \cos \hat{\nu} t \\
\tilde{h}_T(k, t) &= -\frac{i\sigma H}{\hat{\nu}}\tilde{u}_0 \sin \hat{\nu} t - \frac{i\sigma H f}{\hat{\nu}^2}\left\{\tilde{v}_0 - \frac{ikg}{f}\tilde{h}_0\right\} \cos \hat{\nu} t,
\end{aligned} \quad (32)$$

is algebraically extremely complicated, although quite straightforward. However, the salient features are relatively easily treated, and yield the most significant results on the transient behavior. Therefore we only present an outline of the details.

In computing the inversion integrals for (32), terms of the form

$$\int_{-\infty}^{\infty} A(k)\, e^{i(kx \pm \hat{\nu} t)}\, dk$$

arise. The stationary phase points of these integrals arise as the solutions of

$$\phi'(k) = x \pm \frac{f\lambda^2 t \cos k\,\Delta x\,[\sin k\,\Delta x/\Delta x]}{[1 + (\lambda/\Delta x)^2 \sin^2 k\,\Delta x]^{1/2}} = 0. \quad (33)$$

This expression can be simplified by adding $(-x)$ to both sides, squaring and writing $\cos^2 k\,\Delta x$ as $(1 - \sin^2 k\,\Delta x)$, to yield a quadratic in $\sin^2 k\,\Delta x$. The quadratic formula then yields as the points of stationary phase the solutions of

$$\sin^2 k\,\Delta x = \frac{(\lambda^2 f^2 t^2 - x^2) \pm [(\lambda^2 f^2 t^2 - x^2)^2 - 4f^2 t^2 x^2 (\Delta x)^2]^{1/2}}{2\lambda^2 f^2 t^2}. \quad (34)$$

It is then easily shown that as $t \to \infty$, these solutions closely approximate

$$\sin^2 k\,\Delta x \doteq 1 - [x^2/(\lambda^2 f^2 t^2)] \tag{35a}$$

and

$$\sin^2 k\,\Delta x \doteq [x^2(\Delta x)^2 \lambda^{-2}/(\lambda^2 f^2 t^2 - x^2)]. \tag{35b}$$

Clearly, for large $t$, (35b) yields stationary points near both

$$k = [\pm x\lambda^{-1}/(\lambda^2 f^2 t^2 - x^2)^{1/2}]$$

and

$$k = \pm(\pi/\Delta x).$$

(All other solutions of (35b) are beyond the Nyquist cutoff.) The first points are slight variants of the stationary points for the differential case. The points near $\pm(\pi/\Delta x)$ arise solely from the discretization, not the physics of the problem. However, since these points give a behavior of $[(\sin k\,\Delta x)/\Delta x]$ identical to that of the stationary points near $k = 0$, it is easily shown that they contribute computational transients with precisely the same asymptotic behavior as the physical transients.

Note, before we consider the effect of terms introduced by the first solution, (35a), that a necessary and sufficient condition for the stationary phase points to be on the real axis is that the quantity under the radical sign in (34) be positive. After some manipulation, this condition reduces to

$$|\, x\,|/t > f\{[(\Delta x)^2 + \lambda^2]^{1/2} - \Delta x\}, \qquad t > 0.$$

It is not coincidental that the quantity on the right-hand side of the inequality is precisely the group propagation velocity for this quasi-discrete case.

Referring again to the contribution from the stationary phase points satisfying (35a), observe that we can easily show from (33) that

$$\phi''(k) = \frac{\pm\lambda^2 f t}{[1 + (\lambda/\Delta x)^2 \sin^2 k\,\Delta x]^{3/2}} \times \left\{ -\left(1 + \left(\frac{\lambda}{\Delta x}\right)^2 \sin^2 k\,\Delta x\right) \sin^2 k\,\Delta x + \cos^2 k\,\Delta x \right\},$$

and so, near these stationary points,

$$\phi''(k) \sim \pm\lambda^2 f t\,\Delta x/[(\Delta x)^2 + \lambda^2]^{1/2}.$$

Furthermore, note that near these same points

$$\hat{\nu}/f \sim [(\Delta x)^2 + \lambda^2]^{1/2}/\Delta x \qquad \text{and} \qquad \sigma \sim \pm 1/\Delta x.$$

Thus following again the argument of [5, (3.7.5)], we see that the points of stationary phase which arise from the solution to the first lead to transients whose amplitudes, asymptotically, go as follows:

$$\tilde{u}_0 \cos \hat{\nu} t \to \left[\frac{[(\Delta x)^2 + \lambda^2]^{1/2}}{2\pi\lambda^2 f t\, \Delta x}\right]^{1/2} \left|\tilde{u}\left(\pm\frac{\pi}{2\Delta x}, 0\right)\right|;$$

$$\frac{f}{\hat{\nu}}\left\{\tilde{v}_0 - \frac{i\sigma g}{f}\tilde{h}_0\right\} \sin \hat{\nu} t \to \left[\frac{\Delta x}{2\pi\lambda^2 f t[(\Delta x)^2 + \lambda^2]^{1/2}}\right]^{1/2} \left|\tilde{d}\left(\pm\frac{\pi}{2\Delta x}, 0\right)\right|;$$

$$-\frac{f}{\hat{\nu}}\tilde{u}_0 \sin \hat{\nu} t \to \left[\frac{\Delta x}{2\pi\lambda^2 f t[(\Delta x)^2 + \lambda^2]^{1/2}}\right]^{1/2} \left|\tilde{u}\left(\pm\frac{\pi}{2\Delta x}, 0\right)\right|;$$

$$\frac{f^2}{\hat{\nu}^2}\left\{\tilde{v}_0 - \frac{i\sigma g}{f}\tilde{h}_0\right\} \cos \hat{\nu} t \to \left[\frac{(\Delta x)^3}{2\pi\lambda^2 f t((\Delta x)^2 + \lambda^2)^{3/2}}\right]^{1/2} \left|\tilde{d}\left(\pm\frac{\pi}{2\Delta x}, 0\right)\right|;$$

$$-\frac{i\sigma H}{\hat{\nu}}\tilde{u}_0 \sin \hat{\nu} t \to \left[\frac{H^2 f}{2\pi\lambda^2 t\, \Delta x\, [(\Delta x)^2 + \lambda^2]^{1/2}}\right]^{1/2} \left|\tilde{u}\left(\pm\frac{\pi}{2\Delta x}, 0\right)\right|;$$

$$-\frac{i\sigma H f}{\hat{\nu}^2}\left\{\tilde{v}_0 - \frac{i\sigma g}{f}\tilde{h}_0\right\} \cos \hat{\nu} t \to \left[\frac{H^2 f\, \Delta x}{2\pi\lambda^2 t((\Delta x)^2 + \lambda^2)^{3/2}}\right]^{1/2} \left|\tilde{d}\left(\pm\frac{\pi}{2\Delta x}, 0\right)\right|.$$

(Note the evaluations are at $\pm(\pi/2\Delta x)$ since, as $t \to \infty$, these are the only solutions of (35a) that also satisfy the Nyquist limit.)

Viewing the above, it is now clear that the additional stationary points which arise in this quasi-discrete case, and which tend toward $\pm(\pi/2\Delta x)$ as $t \to \infty$ cause two noticeable effects on the transient behavior:

(1) All of these transients now die off as $t^{-1/2}$ due to dispersion. Comparing this to the results in the differential case, we see that these transients are at least as persistent as the differential transients, and for $h(x, t)$ more so, since the differential transients in $h(x, t)$ die out as $t^{-3/2}$.

(2) Two of these transients (the first and fifth) are somewhat ill behaved as $(\Delta x) \to 0$. Assuming $u(x, 0)$ has only finite power, then the $\Delta x$ term in the denominator should be controlled by tail-off of $\tilde{u}(\pm(\pi/2\Delta x), 0)$ as $(\Delta x) \to 0$, however, these terms are virtually certain to be the slowest decaying for small $\Delta x$.

(We note that (35b) also causes stationary points to arise, that tend toward $\pm(\pi/\Delta x)$ as $t \to \infty$; however, the transients from these points do not have an asymptotic dispersion decay that depends on $(\Delta x)$, and decay at the same rate as the differential transients.) Although we shall not show it analytically, we suspect the additional stationary points arise in the quasi-discrete case from two causes, the "folding" in temporal frequency that occurs at $k = \pm(\pi/2\Delta x)$, and the high-frequency cutoff at $k = \pm(\pi/\Delta x)$.

## 4. Conclusion

In this paper we have examined the effect of second-order centered spatial discretization on a dispersive wave equation. We have shown that the methods of Fourier transforms, and in particular the method of stationary phase, are quite useful in such investigations. In both the differential and quasi-discrete cases we have provided closed form expressions for the steady-state solutions. These expressions show that the contribution from any point in the initial state to the final state decays exponentially with distance from that point in both cases, however, the rate of decay is decreased by discretization. The transients in both cases have been analyzed by the method of stationary phase. This analysis shows that the discretization introduces stationary phase points that have no counterpart in the differential case, and, furthermore, these points contribute transients that decay no faster than the differential ones, and in one instance, the discrete transient will dominate the differential transient.

## Appendix 1

Consider the asymptotic behavior of

$$(1/2\pi) \int_{-\infty}^{\infty} A(k) \cos \nu t \, e^{ikx} \, dk = \psi(x, t), \tag{1.1}$$

where $\nu = f(1 + \lambda^2 k^2)^{1/2}$. This integral can be decomposed into

$$2\pi\psi(x, t) = \frac{1}{2} \left\{ \int_{-\infty}^{\infty} A(k) \, e^{i\phi_1(k)} \, dk + \int_{-\infty}^{\infty} A(k) \, e^{i\phi_2(k)} \, dk \right\},$$

where

$$\phi_1(k) = kx + \nu t \qquad \text{and} \qquad \phi_2(k) = kx - \nu t.$$

We can determine the asymptotic behavior as $t \to \infty$ for fixed $x$ of this term by using the method of stationary phase. Let $k_1$ and $k_2$ be defined by

$$\phi_1'(k_1) = \phi_2'(k_2) = 0,$$

where the primes denote derivatives with respect to $k$. But

$$\phi_1'(k) = x + \lambda^2 ftk/(1 + \lambda^2 k^2)^{1/2},$$

and so we find as the point of stationary phase for the first integral:

$$k_1 = -\frac{x/\lambda}{(\lambda^2 f^2 t^2 - x^2)^{1/2}}, \tag{1.2}$$

which is on the path of integration, for $t > x/\lambda f$. Similarly, $\phi_2'(k_2) = 0$ yields as the stationary point for the second integral,

$$k_2 = \frac{x/\lambda}{(\lambda^2 f^2 t^2 - x^2)^{1/2}}, \tag{1.3}$$

which is also on the path of integration for $t > x/\lambda f$.

Observe that

$$\phi_1(k_1) = -\phi_2(k_2) = (1/\lambda)(\lambda^2 f^2 t^2 - x^2)^{1/2},$$

and that

$$\phi_1''(k_1) = -\phi_2''(k_2) = \frac{\lambda^2 f t}{(1 + \lambda^2 k_1{}^2)^{1/2}} = \frac{(\lambda^2 f^2 t^2 - x^2)^{3/2}}{\lambda f^2 t^2} > 0.$$

(Note $|\phi_1''(k_1)| = |\phi_2''(k_2)| = \phi_1''(k_1)$, for $\lambda f t > |x|$, and $\phi'' = d^2\phi/dk^2$.)

Thus we have, by [5, (3.7.5), p. 51],

$$\begin{aligned} 2\pi\psi &\sim \frac{1}{2}\left(\frac{2\pi}{\phi_1''(k_1)}\right)^{1/2} \\ &\quad \times \{A(k_1)\, e^{i[(1/\lambda)(\lambda^2 f^2 t^2 - x^2)^{1/2} + (\pi/4)]} + A(k_2)\, e^{-i[(1/\lambda)(\lambda^2 f^2 t^2 - x^2)^{1/2} + (\pi/4)]}\} \\ &= \frac{1}{2}\left(\frac{2\pi\lambda f^2 t^2}{[\lambda^2 f^2 t^2 - x^2]^{3/2}}\right)^{1/2} \\ &\quad \times \{A(-k_2)\, e^{i[(1/\lambda)(\lambda^2 f^2 t^2 - x^2)^{1/2} + (\pi/4)]} + A(k_2)\, e^{-i[(1/\lambda)(\lambda^2 f^2 t^2 - x^2)^{1/2} + (\pi/4)]}\}. \end{aligned}$$

But note that if $A(k)$ is the transform of a purely real-valued function,

$$A(-k) = \int_{-\infty}^{\infty} a(x)\, e^{ikx}\, dx = A^*(k).$$

Thus, we can write

$$\begin{aligned} \psi &\sim \left[\frac{\lambda f^2 t^2}{2\pi(\lambda^2 f^2 t^2 - x^2)^{3/2}}\right]^{1/2} \Big\{\operatorname{Re} A(k_2) \cos\left[\frac{1}{\lambda}(\lambda^2 f^2 t^2 - x^2)^{1/2} + \frac{\pi}{4}\right] \\ &\quad + \operatorname{Im} A(k_2) \sin\left[\frac{1}{\lambda}(\lambda^2 f^2 t^2 - x^2)^{1/2} + \frac{\pi}{4}\right]\Big\}, \end{aligned} \tag{1.4}$$

where $k_2$ is given by (1.3). Observe for $\lambda f t \gg |x|$, the lead term acts as

$$[1/(2\pi\lambda^2 f t)]^{1/2}.$$

Similarly, we can show

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A(k)\sin \nu t\, e^{ikx}\,dk$$

$$\sim \left[\frac{\lambda f^2t^2}{2\pi(\lambda^2f^2t^2 - x^2)^{3/2}}\right]^{1/2} \Big\{\operatorname{Re} A(k_2)\sin\left[\frac{1}{\lambda}(\lambda^2f^2t^2 - x^2)^{1/2} + \frac{\pi}{4}\right]$$

$$- \operatorname{Im} A(k_2)\cos\left[\frac{1}{\lambda}(\lambda^2f^2t^2 - x^2)^{1/2} + \frac{\pi}{4}\right]\Big\}. \tag{1.5}$$

Lastly, since the trigonometric functions in both expressions have the same frequency, and since

$$\{[\operatorname{Re} A(k)]^2 + [\operatorname{Im} A(k)]^2\}^{1/2} = |A(k)|,$$

we see

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A(k)\cos \nu t\, e^{ikx}\,dk$$

$$\sim \left[\frac{\lambda f^2t^2}{2\pi(\lambda^2f^2t^2 - x^2)^{3/2}}\right]^{1/2} |A(k_2)|\cos\left[\frac{1}{\lambda}(\lambda^2f^2t^2 - x^2)^{1/2} + \varphi_k\right]$$

$$\varphi_k = \frac{\pi}{4} - \tan^{-1}[\operatorname{Im} A(k_2)/\operatorname{Re} A(k_2)]$$

$$= \frac{\pi}{4} - \arg(A(k_2))$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A(k)\sin \nu t\, e^{ikx}\,dk$$

$$\sim \left[\frac{\lambda f^2t^2}{2\pi(\lambda^2f^2t^2 - x^2)^{3/2}}\right]^{1/2} |A(k_2)|\sin\left[\frac{1}{\lambda}(\lambda^2f^2t^2 - x^2)^{1/2} + \varphi_k\right].$$

## Appendix 2

Consider

$$(1/2\pi)\int_{-\infty}^{\infty} [e^{ikx}/(1 + \lambda^2\sigma^2)]\,dk, \qquad \text{where} \quad \sigma = \sin k\,\Delta x/\Delta x. \tag{2.1}$$

The denominator,

$$\psi(k) = 1 + (\lambda/\Delta x)^2 \sin^2 k\,\Delta x \tag{2.2}$$

is an entire function of $k$ in the complex plane. The zeros of $\psi(k)$ are solutions of

$$\sin k\,\Delta x = \pm i(\Delta x/\lambda),$$

or, letting $k = k_r + ik_i$,

$$\sin(k_r\,\Delta x + ik_i\,\Delta x) = \sin(k_r\,\Delta x)\cosh(k_i\,\Delta x) + i\cos(k_r\,\Delta x)\sinh(k_i\,\Delta x)$$
$$= \pm i(\Delta x/\lambda).$$

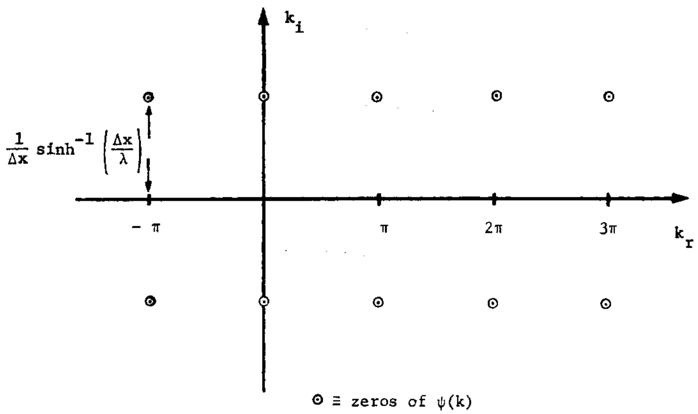
Thus

$$\sin(k_r\,\Delta x)\cosh(k_i\,\Delta x) = 0,$$
$$\cos(k_r\,\Delta x)\sinh(k_i\,\Delta x) = \pm(\Delta x/\lambda).$$

Thus

$$k_r = \pm(n\pi/\Delta x), \qquad k_i = \pm(1/\Delta x)\sinh^{-1}(\Delta x/\lambda), \tag{2.3}$$

as opposed to the poles at $k_r = 0$, $k_i = \pm 1/\lambda$ for the continuous case. Thus all zeros of the denominator lie off the real axis. In fact, they are specifically distributed as shown:




Observe further that at its zeros,

$$\psi'(k) = 2(\lambda^2/\Delta x)\sin k\,\Delta x\cos k\,\Delta x$$
$$= \pm 2(\lambda^2/\Delta x)[i(\Delta x/\lambda)]\cosh[\pm\sinh^{-1}(\Delta x/\lambda)]$$
$$= \pm 2i[\lambda^2 + (\Delta x)^2]^{1/2}, \tag{2.4}$$

where the positive sign holds for $k_i > 0$, and the negative one for $k_i < 0$. Thus the zeros of $\psi(k)$ are simple poles of the integrand.

Consider (2.1) for $x > 0$. (Observe that (2.1) is an even function of $x$.) To ensure $\mathrm{Re}(ikx) < 0$ we will close the contour in $\mathrm{Im}(k) > 0$ half-plane. As long as we route the contour to avoid all the poles there, the integrand is exponentially decaying as $|k| \to \infty$, $\mathrm{Im}(k) > 0$. Thus, we invoke Jordan's lemma to yield, for $x > 0$,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}\,dk}{1+\lambda^2\sigma^2} &= i\sum_{\mathrm{Im}(k)>0}\text{Residues}\left(\frac{e^{ikx}}{1+\lambda^2\sigma^2}\right)\\ &= i\sum_{\mathrm{Im}(k)>0}\frac{e^{ikx}}{\psi'(\hat{k})}, \qquad \hat{k} = k_r + ik_i\\ &= \frac{1}{2[\lambda^2+(\Delta x)^2]^{1/2}}\sum_{-\infty}^{\infty}\exp\left\{ix\left(\frac{n\pi}{\Delta x} + i\frac{1}{\Delta x}\sinh^{-1}\left(\frac{\Delta x}{\lambda}\right)\right)\right.\\ &= \frac{1}{2[\lambda^2+(\Delta x)^2]^{1/2}}\, e^{-\beta x/\lambda}\sum_{-\infty}^{\infty} e^{i(n\pi x/\Delta x)},\end{aligned} \tag{2.5}$$

where

$$\beta = (\lambda/\Delta x)\sinh^{-1}(\Delta x/\lambda). \tag{2.6}$$

Observe that (2.5) is not a convergent series in the usual sense. However, viewed as a generalized function, it is a Fourier series for the function, periodic of period $2\Delta x$, given in the interval $-\Delta x < x < \Delta x$ by

$$S(x) = \sum_{-\infty}^{\infty} c_n e^{i(n\pi x/\Delta x)}, \qquad \text{where} \quad c_n \equiv 1.$$

But observe that

$$c_n = (1/2\Delta x)\int_{-\Delta x}^{\Delta x} S(r)\, e^{-i(n\pi r/\Delta x)}\, dr = 1, \qquad \text{all } n.$$

Thus

$$S(t) = 2\Delta x\,\delta(t),$$

and so its periodic extension becomes

$$S(x) = 2\Delta x\sum_{-\infty}^{\infty}\delta(x - 2n\,\Delta x).$$

Thus

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{1+\lambda^2\sigma^2}\,dk = \frac{\Delta x}{[\lambda^2+(\Delta x)^2]^{1/2}}\, e^{-\beta|x|/\lambda}\sum_{-\infty}^{\infty}\delta(x - 2n\,\Delta x). \tag{2.7}$$

Note, if we view $\beta$ as a function

$$\beta(x) = (1/x)\sinh^{-1} x,$$

it is easily shown that

$$\beta(0) = 1 \qquad \text{and} \qquad \beta'(x) < 0.$$

Thus $0 < \beta(x) \leqslant 1$ with equality only at $x = 0$.

Since, for $\operatorname{Im} k > 0$ and large

$$\frac{i\sigma}{1 + \lambda^2\sigma^2} \sim \frac{\Delta x}{\lambda^2 \sin kx} \sim \frac{\Delta x}{\lambda^2 e^{x \operatorname{Im} k}},$$

the above argument can be essentially repeated for

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i\sigma e^{ikx}}{1+\lambda^2\sigma^2}\,dk &= i\sum_{\operatorname{Im}(\hat{k})>0}\left\{\frac{i\sigma(\hat{k})\,e^{i\hat{k}x}}{\psi'(\hat{k})}\right\}\\ &= \frac{-1}{2\lambda[\lambda^2+(\Delta x)^2]^{1/2}}\,e^{-\beta x/\lambda}\sum_{-\infty}^{\infty}(-1)^n\,e^{i(n\pi x/\Delta x)}, \qquad x > 0\\ &= \frac{-1}{2[\lambda^2+(\Delta x)^2]^{1/2}}\,e^{-\beta|x|/\lambda}\sum_{-\infty}^{\infty}e^{i(n\pi/\Delta x)(x+\Delta x)}\\ &= \frac{-(\Delta x)}{\lambda[\lambda^2+(\Delta x)^2]^{1/2}}\,e^{-\beta|x|/\lambda}\sum_{-\infty}^{\infty}\delta(x-(2n-1)\,\Delta x). \qquad (2.8)\end{aligned}$$

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